William Archer, M.R.I.A.
James Risdon Bennett, M.D.
Dietrich Brandis, Ph.D., F.L.S.
James Caird, C.B.
Prof. John Casey, LL.D.
August Dupré, Ph.D., F.C.S.
James Geikie, F.R.S.E.
James Whitbread Lee Glaisher, M.A.
John Baboneau Nickterlien Hennessey, F.R.A.S.
Emanuel Klein, M.D.
E. Ray Lankester, M.A.
George Strong Nares, Capt. R.N.
Robert Stirling Newall, F.R.A.S.
William Chandler Roberts, F.C.S.
Major-General Henry Y. D. Scott, R.E., C.B.

Thanks were given to the Scrutators.

June 10, 1875.

JOSEPH DALTON HOOKER, C.B., President, in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

Dr. James Risdon Bennett, Mr. James Caird, Mr. James Whitbread Lee Glaisher, Mr. J. Baboneau Nickterlien Hennessey, Mr. William Chandler Roberts, and Major-General Henry Young Darracote Scott were admitted into the Society.

The following Papers were read:—

I. "A Memoir on Prepotentials." By Prof. CAYLEY, F.R.S.

Received April 8, 1875.

(Abstract.)

The present memoir relates to multiple integrals expressed in terms of the $(s+1)$ ultimately disappearing variables $(x\,..\,z, w)$, and the same number of parameters $(a\,..\,c, e)$, and being of the form

$$\int \frac{\rho\, d\varpi}{\{(a-x)^2\,..\,+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

where ρ and $d\varpi$ depend only on the variables $(x\,..\,z, w)$. Such an integral, in regard to the index $\frac{1}{2}s+q$, is said to be "prepotential," and in the particular case $q=-\frac{1}{2}$ to be "potential."

I use throughout the language of hyper-tridimensional geometry: $(x\,..\,z, w)$ and $(a\,..\,c, e)$ are regarded as coordinates of points in $(s+1)$-dimensional space, the former of them determining the position of an element $\rho d\varpi$ of attracting matter, the latter being the attracted point; viz. we have a mass of matter $=\int \rho d\varpi$ distributed in such manner that $d\varpi$ being the element of $(s+1)$- or lower-dimensional volume at the

The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to
Proceedings of the Royal Society of London.
www.jstor.org

point $(x\,..\,z, w)$, the corresponding density is ρ, a given function of $(x\,...\,z, w)$, and that the element of mass $\rho d\varpi$ exerts on the attracted point $(a\,..\,c, e)$ a force proportional to the $(s+2q+1)$th power of the distance $\{(a-x)^2\,..\,+(c-z)^2+(e-w)^2\}^{\frac{1}{2}}$. The integration is extended so as to include the whole attracting mass $\int\rho d\varpi$; and the integral is then said to represent the prepotential of the mass in regard to the point $(a\,..\,c, e)$. In the particular case $s=2$, $q=-\frac{1}{2}$, the force is as the inverse square of the distance, and the integral represents the potential in the ordinary sense of the word.

The element of volume $d\varpi$ is usually either the element of solid (spatial or $(s+1)$-dimensional) volume $dx\,...\,dz\,dw$, or else the element of superficial (s-dimensional) volume $d\mathrm{S}$. In particular, when the surface (s-dimensional locus) is the (s-dimensional) plane $w=0$, the superficial element $d\mathrm{S}$ is $=dx\,...\,dz$. The cases of a less-than-s-dimensional volume are in the present memoir considered only incidentally. It is scarcely necessary to remark that the notion of density is dependent on the dimensionality of the element of volume $d\varpi$: in passing from a spatial distribution, $\rho\,dx\,...\,dz\,dw$, to a superficial distribution, $\rho\,d\mathrm{S}$, we alter the signification of ρ. In fact if, in order to connect the two, we imagine the spatial distribution as made over an indefinitely thin layer or stratum bounded by the surface, so that at any element $d\mathrm{S}$ of the surface the normal thickness is $d\nu$, where $d\nu$ is a function of the coordinates $(x\,..\,z, w)$ of the element $d\mathrm{S}$, the spatial element is $=d\nu\,d\mathrm{S}$, and the element of mass $\rho\,dx\,..\,dz\,dw$ is $=\rho\,d\nu\,d\mathrm{S}$; and then changing the signification of ρ, so as to denote by it the product $\rho\,d\nu$, the expression for the element of mass becomes $\rho\,d\mathrm{S}$, which is the formula in the case of the superficial distribution.

The space or surface over which the distribution extends may be spoken of as the material space or surface; so that the density ρ is not $=0$ for any finite portion of the material space or surface; and if the distribution be such that the density becomes $=0$ for any point or locus of the material space or surface, then such point or locus, considered as an infinitesimal portion of space or surface, may be excluded from and regarded as not belonging to the material space or surface. It is allowable, and frequently convenient, to regard ρ as a discontinuous function, having its proper value within the material space or surface, and having elsewhere the value $=0$; and this being so, the integration may be regarded as extending as far as we please beyond the material space or surface (but so always as to include the whole of the material space or surface)—for instance, in the case of a spatial distribution, over the whole $(s+1)$-dimensional space; and in the case of a superficial distribution, over the whole of the s-dimensional surface of which the material surface is a part.

In all cases of surface-integrals it is, unless the contrary is expressly stated, assumed that the attracted point does not lie on the material surface; to make it do so is, in fact, a particular supposition. As to solid integrals, the cases where the attracted point is not, and is, in the material space may be regarded as cases of coordinate generality; or we may regard the latter one as the general case, deducing the former one from it by supposing the density at the attracted point to become $=0$.

The present memoir has chiefly reference to three principal cases, which I call A, C, D, and a special case, B, included both under A and C: viz. these are:—

A. The prepotential-plane case; q general, but the surface is here the plane $w=0$, so that the integral is

$$\int \frac{\rho\, dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}}.$$

B. The potential-plane case; $q=-\frac{1}{2}$, and the surface the plane $w=0$, so that the integral is

$$\int \frac{\rho\, dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

C. The potential-surface case; $q=-\frac{1}{2}$, the surface arbitrary, so that the integral is

$$\int \frac{\rho\, dS}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

D. The potential-solid case; $q=-\frac{1}{2}$, and the integral is

$$\int \frac{\rho\, dx \ldots dz\, dw}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

It is, in fact, only the prepotential-plane case which is connected with the partial differential equation

$$\left(\frac{d^2}{da^2} \ldots + \frac{d^2}{dc^2} + \frac{d^2}{de^2} + \frac{2q+1}{e}\frac{d}{de}\right)\mathrm{V}=0,$$

considered in Green's memoir 'On the Attractions of Ellipsoids' (1835), and called here the prepotential equation. For this equation is satisfied by the function

$$\frac{1}{\{a^2 \ldots + c^2 + e^2\}^{\frac{1}{2}s+q}},$$

and therefore also by

$$\frac{1}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}},$$

and consequently by the integral

$$\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}, \quad \ldots\ldots \quad \text{(A)}$$

that is by the prepotential-plane integral; but the equation is *not* satisfied by the value

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

nor, therefore, by the prepotential-solid, or general superficial, integral.

But if $q=-\frac{1}{2}$, then, instead of the prepotential equation, we have

$$\left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V=0\,;$$

and this is satisfied by

$$\frac{1}{\{a^2\ldots+c^2+e^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

and therefore also by

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

Hence it is satisfied by

$$\int\frac{\rho\,dx\ldots dz\,dw}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}, \quad \ldots\ldots \quad \text{(D)}$$

the potential-solid integral, *provided that the point* $(a\ldots c, e)$ *does not lie within the material space*: I would rather say that the integral does *not* satisfy the equation, but of this more hereafter; and it is satisfied by

$$\int\frac{\rho\,dS}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}, \quad \ldots\ldots \quad \text{(C)}$$

the potential-surface integral. The potential-plane integral (B), as a particular case of (C), of course also satisfies the equation.

Each of the four cases gives rise to what may be called a distribution-theorem, viz. given V a function of $(a\ldots c, e)$ satisfying certain prescribed conditions, but otherwise arbitrary, then the form of the theorem is that there exists and can be found an expression for ρ, the density or distribution of matter over the space or surface to which the theorem relates, such that the corresponding integral V has its given value, viz. that in A and B there exists such a distribution over the plane $w=0$, in C such a distribution over a given surface, and in D such a distribution in space. The establishment, and exhibition in connexion with each other, of these four distribution-theorems is the principal object of the present memoir; but the memoir contains other investigations which have presented themselves to me in treating the question. It is to be noticed that the theorem A belongs to Green, being in fact the fundamental theorem of his memoir of 1835, already referred to. Theorem C, in the particular case of tridimensional space, belongs also to him, being given in his 'Essay on the Application of Mathematical Analysis to the theories of Electricity and Magnetism' (Nottingham, 1828), being partially rediscovered by Gauss in the year 1840; and theorem D, in the same case of tridimensional space, to Lejeune-Dirichlet: see his memoir "Sur un moyen général de verifier l'expression du potential relatif à une masse quelconque homogène ou héterogène," Crelle, t. xxxii. pp. 80–84 (1846). I refer more particularly to these and other researches by Gauss, Jacobi, and others in the course of the present memoir.

"On the Fossil Mammals of Australia.—Part X. Family MACROPODIDÆ: Mandibular Dentition and Parts of the Skeleton of *Palorchestes,* with additional evidences of *Sthenurus, Macropus Titan,* and *Procoptodon.*" By Professor OWEN, C.B., F.R.S. Received May 10, 1875.

(Abstract.)

In this "Part" the author gives additional evidences of extinct genera and species of Kangaroos defined in the two preceding Parts (VIII. and IX.). To the *Palorchestes Azaël* he adds characters of the mandible and mandibular teeth, and gives a restoration of the entire skull; the pelvis, femur, tibia, calcaneum, and principal bones of the hind foot of this gigantic species are described and figured.

Of *Macropus Titan* the author restores the entire skull and femur. Of *Sthenurus Atlas* he describes and figures the incisor teeth, the deciduous dentition, and the fore part of the skull of a young individual: of the larger species of this genus, *Sthenurus Brehus,* the entire skull and dentition are restored. The "Part" concludes with the restoration of certain bones of the hind foot in a Kangaroo slightly exceeding the largest *Macropus major* in size (indicated as a *Macropus affinis*), in the *Phascolagus altus,* in *Palorchestes Azael,* and in the three species of *Procoptodon* (*Pusio, Rapha,* and *Goliah*). The paper concludes with remarks on the transitionary character of the latter forms, as bridging the gap between the saltigrade and gravigrade groups of phytiphagous Marsupialia.

The paper is illustrated by subjects for thirteen plates.